*Duo*limpiadi II

Searching for the n-gon tablet

2026-02-22

> **Lore**
> While Madeline and Biagio were researching the ancient problems from Duolimpiadi I, they discovered about the equally ancient tale of the n-agon tablet. It is said that reading it gives the ability to be decent in geometry. Hopeful that the tale is true, they decide to start a quest to find the tablet.

II^0. The game and the Game

> **Lore**
> Biagio loves making Madeline lose the Game. Before leaving, then, he comes up with a plan: he proposes her two games, the first of which is quite interesting ad the second of which is a trap. Will he succeed in his goal?

Biagio proposes two games to Madeline. The first is the following: he chooses two sequences of positive integers $(a_i)_{i=1}^m$ e $(b_j)_{j=1}^n$ and creates a grid with m rows and n columns, where in each cell (i, j) he writes the number $a_i \cdot b_j$. He shows the grid to Madeline, who has to guess the two sequences. For which grids created by Biagio does there exist a strategy for Madeline that gurantees her the win?[1]

II^1. Work smarter, not harder

> **Lore**
> It's said that the tablet is in one of ten locations, all contained in an equilateral triangle with side length $2\mathrm{Mm}$. Madeline is lazy and doesn't want to travel too much. Therefore, she wonders if there's two locations that are quite close to each other; if they existed she'd insist to check those two first.

Given an equilateral triangle with side 2, ten points are chosen in its interior. Determine if there always exist two points among them that are distant at most $\dfrac{2^{2^2}}{2^{2^2} + 2^2 + 2}$.

[1]The second game is the one you just lost.

$\mathrm{II}^{\log_{\mathrm{II}}(3)}$. The infinite campground

Lore

Biagio and Madeline have checked nine of the ten locations, and have almost lost all hope. While traveling towards the tenth, they decide to stop in a campground to rest. Arriving from above, they see that all the tents are laid out in rows. Madeline notices that there's infinite rows, but Biagio adds that every row contains a finite number of tents. Madeline then suggests, «That's true, in fact there's exactly $\mathrm{Duo}(n)/\mathrm{Duo}(0)$ tents in every row!» Biagio answers, «is that number actually always natural?», to which she replies, «have you ever seen half a tent?»

Call $\mathrm{Duo}(n)$ the series $\sum_{j=0}^{\infty} \frac{j^n}{j!}$, which may be assumed to converge. Prove that

$$\frac{\mathrm{Duo}(n)}{\mathrm{Duo}(0)} \in \mathbb{N}$$

for all $n \in \mathbb{N}$.

$\mathrm{II}^{\mathrm{II}}$. Just one ■ to be done

Lore

Madeline and Biagio arrived at the tenth location. At the entrance they see a problem carved on a door. To the side there's some sheets of paper, pencils and crayons. Below the problem is a rectangular slot, it's clear that they should solve the problem and insert the proof. Madeline suggests this obstacle is promising for finding the tablet, but Biagio notices that it doesn't say anywhere that behind the large closed door is actually what they're looking for.

Let $\sigma_0(n)$ be the number of positive divisors of n. Find all the $n \in \mathbb{Z}^+$ for which

$$\sigma_0(3^n - \mathbf{2}^n) = n$$